ON THE COLORED HOMFLY-PT, MULTIVARIABLE AND KASHAEV LINK INVARIANTS

NATHAN GEER AND BERTRAND PATUREAU-MIRAND

ABSTRACT. We study various specializations of the colored HOMFLY-PT polynomial. These specializations are used to show that the multivariable link invariants arising from a complex family of $\mathfrak{sl}(m|n)$ super-modules previously defined by the authors contains both the multivariable Alexander polynomial and Kashaev's invariants. We conjecture these multivariable link invariants also specialize to the generalized multivariable Alexander invariants defined by Y. Akutsu, T. Deguchi, and T. Ohtsuki.

CONTENTS

INTRODUCTION

Let $\mathfrak{sl}(m|n)$ be the special linear Lie superalgebra. In [8] the authors show that the Reshetikhin-Turaev quantum invariant of links arising from the category of quantized $\mathfrak{sl}(m,n)$ modules can be modified to produce non-trivial multivariable link invariants for each $m,n \in \mathbb{N}^*$ and $c \in \mathbb{N}^{m+n-2}$. For $m \geq 2$, let $M^0_{\mathfrak{sl}(m|1)}$ be the invariant associated to $\mathfrak{sl}(m|1)$ and $0 \in \mathbb{N}^{m-1}$. If L is a link with k components and $k \geq 2$ then $M^0_{\mathfrak{sl}(m|1)}(L) \in \mathbb{Z}[q^{\pm 1}, q_1^{\pm 1}, \dots, q_k^{\pm 1}]$. If $k = 1$ then $M^0_{\mathfrak{sl}(m|1)}(L) \in g^{-1}\mathbb{Z}[q^{\pm 1}, q_1^{\pm 1}]$ where g is an element of $\mathbb{Z}[q^{\pm 1}, q_1^{\pm 1}]$ which does not depend on L.

The variable q is the usual quantum parameter coming from the Reshetikhin-Turaev quantum group construction. The origin of the variable q_i is as follows. The isomorphism classes of irreducible finite-dimensional $\mathfrak{sl}(m|1)$-module are parameterized by $\mathbb{N}^{m-1} \times \mathbb{C}$. The invariant $M^0_{\mathfrak{sl}(m|1)}$ is constructed by assigning the family of modules corresponding to $(0, ..., 0, \alpha_i)$, for $\alpha_i \in \mathbb{C}$, on the ith component of L. The variable q_i corresponds to q^{α_i} in this construction.

These invariants associate a variable to each component of the link. There are very few known invariants with such properties. Such invariants including the multivariable Alexander polynomial and the generalized multivariable Alexander invariants $\{ADO_m\}_{m \geq 2}$ defined by Akutsu, Deguchi and Ohtsuki [2]. In this paper we use various specializations of the

Date: February 2, 2008.

HOMFLY-PT polynomial to show that the invariants $\{M^0_{\mathfrak{sl}(m|1)}\}_{m\geq 2}$ contain the multivariable Alexander polynomial and Kashaev's invariants [11] (which are a specialization of the invariants ADO_m).

Using the quantum dilogarithm, Kashaev [11] defines a family of complex valued link invariants K_m indexed by integers $m \geq 2$. He then observes that the hyperbolic volume of the complement of some simple hyperbolic knots is determined by the asymptotic behavior of these link invariants and conjectures that this is true for any hyperbolic knot. In [15], H. Murakami and J. Murakami reformulate and strengthen Kashaev's conjecture as follows: "The colored Jones polynomials determine the simplicial volume for any knot." This conjecture has become known as the Volume Conjecture.

In making their reformulation H. Murakami and J. Murakami show that the set of generalized multivariable Alexander invariants $\{ADO_m\}_{m\geq 2}$ and the set of colored Jones polynomials have a non-trivial intersection. Moreover, they show that this intersection contains Kashaev's invariants.

In Section 4 we will show that a similar result holds for the invariants $M^0_{\mathfrak{sl}(m|1)}$; namely, that the intersection of the set of multivariable link invariants $\{M^0_{\mathfrak{sl}(m|1)}\}_{m\geq 2}$ and the set of colored HOMFLY-PT polynomials contains Kashaev's invariants. In the first half of this paper, we extend the fact that the two variable Links-Gould invariants $\{LG^{m,1}\}_{m\geq 2}$ [12] specialize to the Alexander polynomial [4] by showing that the invariants $\{M^0_{\mathfrak{sl}(m|1)}\}$ which can be seen as multivariable extensions of the Links-Gould invariants specialize to the Conway function ∇.

For a link L with k component and any integer $m \geq 2$ the above discussion can be summarized by the following diagram:

$$\begin{tikzcd}
 & & M^0_{\mathfrak{sl}(m|1)}(q,q_1,\dots,q_k) \arrow[dll, "\text{Th } 3;\, t_i=q_i^m"'] \arrow[ddll, "q=\sqrt[m]{-1}"] \arrow[dd, "\text{Th } 1"', "q_i=\tau^{-1}"] \arrow[ddrr, "q=\sqrt[m+1]{-1}"'] \arrow[drr, dotted, "\text{Conj } 4.7;\, q_i=q^{a_i}"] & & \\
\nabla(t_1,\dots,t_k) \arrow[d, "t_i^2=t"] & & & & ADO_{m+1}(a_1,\dots,a_k) \arrow[d, "a_i=m"', "\text{Th } 4\ [15]"] \\
\Delta(t) & & LG^{m|1}(\tau,q) \arrow[ll, "q=\sqrt[m]{-1},\, t=\tau^{2m}"', "\text{Th } 2;\ [4]"] \arrow[rr, "\tau=q=\sqrt[m+1]{-1}", "\text{Th } 5"'] & & K_{m+1}
\end{tikzcd}$$

where a solid arrow represents an equality of the link invariants after a specialization (and possibly a renormalization) and a dotted arrow represents a similar equality which we conjecture to be true. For example, the down arrow in the middle of the diagram means

$$LG^{m,1}(\tau,q) = \left(\prod_{i=0}^{m-1}\left(\tau^{-1}q^i - \tau q^{-i}\right)\right) M_m(q,\tau^{-1},\dots,\tau^{-1})$$

and "Th 1" indicates that the proof of this equality is given in Theorem 1.

The construction of $\{M^0_{\mathfrak{sl}(m|1)}\}$ differs from the construction of the link invariants $\{ADO_m\}$. The definition of $M^0_{\mathfrak{sl}(m|1)}$ relies on ribbon categories whereas the definition of $\{ADO_m\}$ use the Markov trace for the colored braid group. In both case, the standard method using ribbon categories or the Markov trace is trivial. To over come this difficulty, both constructions rely

on a regularization of the corresponding standard method. The dotted arrow in the above diagram is correspond to Conjecture 4.7.

1. The Multivariable Invariant associated with $\mathfrak{sl}(m|1)$

In this section we show that the invariant $M^0_{\mathfrak{sl}(m|1)}$ is related with the Links-Gould invariant and the Conway function.

First, let us formulate a general notation which will be used throughout. Let $\mathcal{C}$ be a category such that the set of endomorphisms of an object V are a module over some ring K. Suppose that f and g are endomorphisms of V such that $f = xg$ where $x \in K$ is a scalar. Then we set $\langle f : g \rangle = x$.

1.1. **$U_h(\mathfrak{g})$-modules.** Here we recall a particular category of modules over the Drinfeld-Jimbo quantization associated to the special linear Lie superalgebra $\mathfrak{sl}(m|n)$. For more details please see [6, 8] and the references within.

Set $\mathfrak{g} = \mathfrak{sl}(m|n)$ $(m \neq n)$ and let $U_h(\mathfrak{g})$ be the DJ quantization associated to $\mathfrak{g}$ over $\mathbb{C}[[h]]$. In this paper, by a $U_h(\mathfrak{g})$-module we mean a topologically free $U_h(\mathfrak{g})$-module of finite rank, i.e. a module over $U_h(\mathfrak{g})$ which is of the form $V[[h]]$ where V is a finite dimensional $\mathfrak{g}$-module. In [6], it is shown that every $\mathfrak{g}$-module V can be deformed to a $U_h(\mathfrak{g})$-module which we denote by $\widetilde{V}$. Let $\mathcal{M} = \mathrm{Mod}_{U_h(\mathfrak{g})}$ be the ribbon category of topologically free $U_h(\mathfrak{g})$-modules of finite rank and let $\mathrm{Mod}_{\mathfrak{g}}$ be the category of finite dimensional $\mathfrak{g}$-modules.

An object $\widetilde{V}$ of $\mathcal{M}$ is irreducible if $End_{U_h(\mathfrak{g})}(\widetilde{V}) = \mathbb{C}[[h]]\,\mathrm{Id}_{\widetilde{V}}$. The classical limit of $\widetilde{V}$ is the $\mathfrak{g}$-module $\widetilde{V}/(h\widetilde{V})$. Denote by $C : \mathrm{Mod}_{U_h(\mathfrak{g})} \longrightarrow \mathrm{Mod}_{\mathfrak{g}}$ the "classical limit" functor. Then $\widetilde{V}$ is a deformation of $V = C(\widetilde{V})$ (unique up to isomorphism).

Lemma 1.1. *The deformation of an irreducible $\mathfrak{g}$-module is an irreducible $U_h(\mathfrak{g})$-module.*

Proof. If $f \in \mathrm{End}_{U_h(\mathfrak{g})}(\widetilde{V})$, the weight decomposition of V is preserved in $\widetilde{V}$ and f respects it. So for a highest weight vector v of $\widetilde{V}$, we have $f(v) = xv$ for some $x \in \mathbb{C}[[h]]$. Then $Ker(f - x\,\mathrm{Id}_{\widetilde{V}})$ is a $U_h(\mathfrak{g})$-module whose classical limit is V implying $Ker(f - x\,\mathrm{Id}_{\widetilde{V}}) = \widetilde{V}$ and so $f = x\,\mathrm{Id}_{\widetilde{V}}$. □

Lemma 1.1 implies that an irreducible $U_h(\mathfrak{g})$-module is determined by the underlying irreducible $\mathfrak{g}$-module. Every irreducible finite-dimensional $\mathfrak{g}$-module has a highest weight $\nu \in \mathfrak{h}^*$ (where $\mathfrak{h}$ is the Cartan sub-superalgebra). We denote such a module by V_ν. The set of isomorphism classes of irreducible finite-dimensional $\mathfrak{g}$-modules are in one to one correspondence with the set of dominant weights. These modules are parameterized by $\mathbb{N}^{m+n-2} \times \mathbb{C}$ and are divided into two classes: typical and atypical. We denote the weight corresponding to $(c, \alpha) \in \mathbb{N}^{m+n-2} \times \mathbb{C}$ as ν^c_α and say $\widetilde{V}$ is typical if V is typical.

1.2. **The invariant $M^c_{\mathfrak{g}}$.** Let $\mathcal{T}_{U_h(\mathfrak{g})}$ be the category of $\mathcal{M}$-colored ribbon graphs (that is the category of framed oriented $\mathcal{M}$-colored graph with coupons labeled by even morphisms in $\mathcal{M}$). Let $F_{m|n}$ be the Reshetikhin-Turaev functor from $\mathcal{T}_{U_h(\mathfrak{g})}$ to $\mathcal{M}$ (see [20]). In [8], the authors introduce a renormalization $F_{m|n}$ and show that this renormalization leads to a multivariable invariant of ordered links $M^c_{\mathfrak{g}}$ for $c \in \mathbb{N}^{m+n-2}$. Let us now briefly recall this construction.

Let V be an irreducible module in $\mathcal{M}$. Let $T_V \in \mathcal{T}_\mathcal{M}$ be a $(1,1)$-ribbon graph whose open edge is oriented down and colored with V. Then $F_{m|n}(T_V) \in \mathrm{End}_\mathcal{M}(V) = \mathbb{C}[[h]]\,\mathrm{Id}_V$. Using the notation at the beginning of this section we have that $F_{m|n}(T_V) = \left\langle F_{m|n}(T_V) \,:\, \mathrm{Id}_V \right\rangle \mathrm{Id}_V$.

For the braid closure $\hat{T}_V$ of T_V (which is also its quantum trace in $\mathcal{T}_\mathcal{M}$) one has $F_{m|n}(\hat{T}_V) = \mathrm{qdim}(V) \left\langle F_{m|n}(T_V) \,:\, \mathrm{Id}_V \right\rangle$ which is zero if V is typical. Hence $F_{m|n}$ is zero on any closed ribbon graph colored by typical modules. In [8], the authors define a modified quantum dimension which is a function d from the set of typical modules to $\mathbb{C}[[h]][h^{-1}]$, which has the property that the mapping $\hat{T}_V \mapsto \mathsf{d}(V) \left\langle F_{m|n}(T_V) \,:\, \mathrm{Id}_V \right\rangle$ is a well defined invariant of $\mathcal{M}$-colored closed ribbon graphs. We denote this invariant by F'.

For a link $L = L_1 \cup \cdots \cup L_k$ whose components are colored by the deformations of typical modules $V_{\nu^c_{\alpha_i}}$ with the same integral weight c but with various complex parameters $\alpha_1, \ldots \alpha_k$, $F'(L)$ depends continuously on these k parameters. In fact it defines an unique rational function $M^c_\mathfrak{g}(L) \in \mathbb{Q}(q, q_1, \ldots, q_k)$ such that

$$F'(L) = f(L).M^c_\mathfrak{g}(L)(q, q^{\alpha_1}, \ldots, q^{\alpha_k}) \tag{1}$$

where $f(L)$ is a function depending only on the linking matrix of L (see [8]). Here we use the notation

$$q = \mathrm{e}^{h/2} \qquad q^x = \mathrm{e}^{xh/2}\ .$$

When L is not a knot (i.e. if $k \geq 2$), $M^c_\mathfrak{g}(L)$ happens to be a Laurent polynomial in these $k+1$ variables (for knots there is a denominator coming from d). The function $M^c_\mathfrak{g}$ is a multivariable invariant of (unframed) ordered oriented links.

For $\mathfrak{g} = \mathfrak{sl}(m|1)$, dominant weights are given by $\mathbb{N}^{m-1} \times \mathbb{C}$ which can be written in the basis $(w_1, \ldots, w_m)$ of the fundamental weights. As above irreducible finite-dimensional $\mathfrak{g}$-modules are denoted $V_{\nu^c_\alpha}$ ($c \in \mathbb{N}^{m-1}$, $\alpha \in \mathbb{C}$) where $V_{\nu^0_\alpha}$ is typical iff $\alpha \notin \{0, -1, \ldots, 1-m\}$.

Let $LG^{m|1}$ be the Links-Gould invariant (see [4] and references within).

Theorem 1.

$$LG^{m|1}(q^{-\alpha}, q) = \left(\prod_{i=0}^{m-1} \left(q^{\alpha+i} - q^{-(\alpha+i)} \right) \right) M^0_{\mathfrak{sl}(m|1)}(q, q^\alpha, \ldots q^\alpha)$$

Proof. In [4] the Links-Gould invariant is computed using an R-matrix of $U_q\mathfrak{gl}(m|1) \simeq U_q\mathbb{T}_1 \otimes U_q\mathfrak{sl}(m|1)$ (isomorphism of Hopf algebras) where $U_q\mathbb{T}_1$ is the (co-)commutative Hopf algebra of polynomials in one primitive variable t. They consider the 2^m-dimensional minimal typical $\mathfrak{sl}(m|1)$-representation $\widetilde{V}_{\nu^0_\alpha}$ with highest weight $\alpha w_m = (0, \ldots, 0, \alpha)$ (for a generic value of α) on which t acts by some scalar. Hence, they compute the Reshetikhin-Turaev invariant of a $(1,1)$-tangle $T_{\widetilde{V}_{\nu^0_\alpha}}$ where each component is colored by $\widetilde{V}_{\nu^0_\alpha}$ with a R-matrix that, differs from ours by a scalar and is rescaled so that the corresponding framed tangle invariant does not depend of its framing. In other words, up to a correction for the framing the Links-Gould invariant of a link L is given by $\left\langle T_{\widetilde{V}_{\nu^0_\alpha}} \,:\, \mathrm{Id}_{\widetilde{V}_{\nu^0_\alpha}} \right\rangle$ where the closer of $T_{\widetilde{V}_{\nu^0_\alpha}}$ is the link L whose components are all colored by $\widetilde{V}_{\nu^0_\alpha}$.

Therefore, with the convention of [4] where $\tau = q^{-\alpha}$, we get that for any link L,

$$LG^{m|1}(L)(\tau, q) = M^0_{\mathfrak{sl}(m|1)}(L)(q, \tau^{-1}, \ldots, \tau^{-1}) / M^0_{\mathfrak{sl}(m|1)}(unknot)(q, \tau^{-1}).$$

The theorem then follows from Lemma 4.8 and the fact that $M^0_{\mathfrak{sl}(m|1)}(unknot) = \mathsf{d}(\widetilde{V}_{\nu^0_\alpha})$. $\square$

Remark 1.2. *Theorem 1 gives another proof that the Links-Gould invariant of a* $(1,1)$*-tangle* T *depends only of the link closure* $\widetilde{T}$ *of* T*. (This is not trivial for tangles with several components).*

In [4] (Theorem 5) the following relation between LG and the Alexander-Conway polynomial is proved:

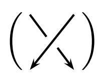
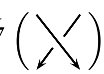
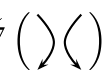
Theorem 2. *(De Wit, Ashii, Links [4]). For all integer* $m \geq 2$*, one has*

$$\Delta(L)(\tau^{2m}) = LG^{m|1}(\tau, \mathrm{e}^{\sqrt{-1}\pi/m})$$

We now prove a generalization of this theorem that shows that the invariants $M^0_{\mathfrak{sl}(m|1)}$ specialize to the multivariable Conway potential function. For this, we use a modified version of Turaev's axioms for the Conway map ([19] section 4) that the authors have proved and used in [7]:

Lemma 1.3. *The Conway function is the map uniquely determined by*

(1) ∇ *assigns to each ordered oriented link* L *in* S^3 *an element of the field* $\mathbb{Q}(t_1, \ldots, t_k)$ *where* k *is the number of components of* L*.*
(2) $\nabla(L)$ *is unchanged under (ambient) isotopy of the link* L*.*
(3) $\nabla(\textit{unknot}) = (t_1 - t_1^{-1})^{-1}$*.*
(4) *If* $k \geq 2$ *then* $\nabla(L) \in \mathbb{Q}[t_1^{\pm 1}, \ldots, t_k^{\pm 1}]$*.*
(5) *The one variable function on links with several components* $\widetilde{\nabla}(L) = \nabla(L)(t, t, \ldots, t) \in \mathbb{Q}[t^{\pm 1}]$ *is unchanged by a reordering of the components of* L*.*
(6) *(Conway Identity)*

$\widetilde{\nabla}$  $-\ \widetilde{\nabla}$  $= (t - t^{-1})\widetilde{\nabla}$ 

(7) *(Modified Doubling Axiom). If* L^+ *(resp.* L^-*) is obtained from the link* $L = L_1 \cup \cdots \cup L_k$ *by replacing the* i^{th} *component* L_i *by its* $(2,1)$*-cable (resp. by its* $(2,-1)$*-cable) then*

$$t_k \nabla(L^+)(t_1, \ldots, t_k) - t_i^{-1}\nabla(L^-)(t_1, \ldots, t_k) =$$
$$\left(\prod_{j \neq i} t_j^{lk_{ij}}\right)(t_i^2 - t_i^{-2})\nabla(L)(t_1, \cdots, t_{i-1}, t_i^2, t_{i+1}, \cdots, t_k)$$

where $(lk_{ij})_{i,j=1\cdots k}$ *is the linking matrix of* L*.*

Theorem 3. *For any oriented link* L *with* k *ordered component, one has*

$$\nabla(L)(q_1^m, \ldots, q_k^m) = \mathrm{e}^{\sqrt{-1}(m-1)\pi/2}\, M^0_{\mathfrak{sl}(m|1)}(L)(\mathrm{e}^{\sqrt{-1}\pi/m}, q_1, \ldots, q_k).$$

Proof. For $i \in \mathbb{N}$, set $t_i = q_i^m$ and $M'(q_1, \ldots, q_k) = \mathrm{e}^{\sqrt{-1}(m-1)\pi/2} . M^0_{\mathfrak{sl}(m|1)}(\mathrm{e}^{\sqrt{-1}\pi/m}, q_1, \ldots, q_k)$ It is clear that M' satisfies the Axioms (1), (2), (4) and (5) of Lemma 1.3 (we neglect the fact that M' leave a priori in the extension $\mathbb{Q}[q_i^{\pm 1}]$ of $\mathbb{Q}[t_i^{\pm 1}]$). Axiom (3) also follows easily from

$$M'(q_1)(\textit{unknot}) = \mathrm{e}^{\sqrt{-1}(m-1)\pi/2} . M^0_{\mathfrak{sl}(m|1)}(\mathrm{e}^{\sqrt{-1}\pi/m}, q_1)(\textit{unknot}) = 1/(q_1^m - q_1^{-m})$$

where the second equality follows from Lemma 4.9.

Let us show that the Conway Identity and the Modified Doubling Axiom hold. To do this we need to recall the following facts. For a link L whose ordered components are colored by $\widetilde{V}_{\nu^0_{a_1}}, \dots, \widetilde{V}_{\nu^0_{a_k}}$, from [8] we have that F' and $M^0_{\mathfrak{sl}(m|1)}$ are related by

$$\begin{aligned} M^0_{\mathfrak{sl}(m|1)}(L)(q, q^{a_1}, \dots, q^{a_k}) &= q^{-\sum lk_{ij} <a_i w_m, a_j w_m + 2\rho>} F'(L) \\ &= q^{m \sum lk_{ij} a_i - r \sum lk_{ij} a_i a_j} F'(L) \end{aligned} \tag{2}$$

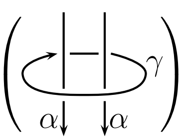
where w_m is the m^{th} fundamental weight, $<,>$ is the symmetric non-degenerate form on the Cartan sub-superalgebra defined in [8], and $r = < w_m, w_m > = \frac{m}{1-m}$. Let us temporarily extend the scalar to its quotient field.

Define $c_i \in \mathbb{N}^{m-1}$ for $0 \le i \le m$ as follows: $c_0 = c_m = (0, ..., 0)$, $c_i = (0, ..., 0, 1, 0, ..., 0)$ where the 1 is in the ith slot, for $1 \le i \le m-1$. One can use character formulas to show that (see [8] (Lemma 2.8)) :

$$\widetilde{V}_{\nu^0_\alpha} \otimes \widetilde{V}_{\nu^0_\alpha} \simeq \bigoplus_{i=0}^{m} \widetilde{V}_i \tag{3}$$

where V_i is the irreducible module with highest weight ν_i which is equal to $\nu^{c_i}_{2\alpha+m-i-1}$ for $0 \le i \le m-1$ and $\nu^{c_m}_{2\alpha}$ for $i = m$. Thus, with the convention that $w_0 = 0$, we have $\nu_i = w_i + (2\alpha + m - i - 1) w_m$.

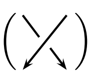
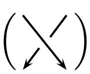
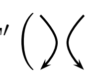
We consider a 2-cable of the long Hopf link whose close component is colored by $\widetilde{V}_{\nu^0_\gamma}$ where $\gamma \in \mathbb{C}$. Its image under F is an endomorphism f of $\widetilde{V}_{\nu^0_\alpha} \otimes \widetilde{V}_{\nu^0_\alpha}$:

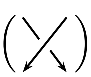
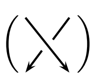
$f \;=\; F\Big($  $\Big)$.

It acts on each simple summand $\widetilde{V}_i$ of $\widetilde{V}_{\nu^0_\alpha} \otimes \widetilde{V}_{\nu^0_\alpha}$ by a scalar f_i. These scalars have been computed in [8] (where they are denoted by $S'(\widetilde{V}_{\nu^0_\gamma}, \widetilde{V}_i)$. In particular, $f_i = \phi_{\nu_i+\rho}(\mathrm{sch}(V_{\nu^0_\gamma}))$ where sch is the super character of $V_{\nu^0_\gamma}$ and $\phi_{\nu_i+\rho}$ is a ring map. In Proposition 1.1 of [8] we show that $\mathrm{sch}(V_{\nu^0_\gamma}) = u^{\gamma w_m} \chi'_1$ where the coefficient of u^a is the dimension of the a-weight space and χ'_1 is a linear combination of elements u^b which are indexed by the set of positive odd roots $\{b\}$. It follows that $f_i = q^{2\gamma(i-m)} q^{r(2\gamma+m-1)} \phi_{\nu_i+\rho}(\chi'_1)$ and $\phi_{\nu_i+\rho}(\chi'_1)$ is a non-zero Laurent polynomial in two variables evaluated at q and $q^{2\alpha}$.

Hence in a ring where the $f_i - f_j$ are invertible, (for example, for irrational values of $\alpha, \gamma \in \mathbb{C}$ when $q = \mathrm{e}^{\sqrt{-1}\pi/m}$), the projector P_i on $\widetilde{V}_i \simeq Im(P_i)$ is realized in $\mathrm{End}(\widetilde{V}_{\nu^0_\alpha} \otimes \widetilde{V}_{\nu^0_\alpha})$ as linear combination of power of f. Hence the decomposition of Equation (3) is still valid for $q = \mathrm{e}^{\sqrt{-1}\pi/m}$.

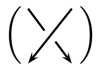
Now any endomorphism g of $\widetilde{V}_{\nu^0_\alpha} \otimes \widetilde{V}_{\nu^0_\alpha}$ acts on $\widetilde{V}_i$ by some scalar g_i. Hence $g = \sum_i g_i P_i$. For the braid closure $\widehat{P}_i$ of P_i, one has $F'(\widehat{P}_i) = \mathsf{d}(\widetilde{V}_i)$ and so $F'(\widehat{g}) = \sum_i g_i \, \mathsf{d}(\widetilde{V}_i)$. For $q = \mathrm{e}^{\sqrt{-1}\pi/m}$ and $0 < j < m$, one has $\mathsf{d}(\widetilde{V}_j) = 0$ (see Lemma 4.10). So modulo the kernel of F', one has, $\dim_{\mathbb{C}}(\mathrm{End}(\widetilde{V}_{\nu^0_\alpha} \otimes \widetilde{V}_{\nu^0_\alpha})) = 2$. Using this, we are going to show that for $q = \mathrm{e}^{\sqrt{-1}\pi/m}$, F' satisfies the following two skein relations:

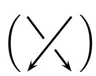
$q^{m\alpha - r\alpha^2} F'\Big($  $\Big) - q^{-m\alpha + r\alpha^2} F'\Big($  $\Big) = (q^{m\alpha} - q^{-m\alpha}) F'\Big($  $\Big)$ (4)

$q^{2m\alpha - r\alpha^2} F'\Big($  $\Big) - q^{-2m\alpha + r\alpha^2} F'\Big($  $\Big) = (q^{2m\alpha} - q^{-2m\alpha}) P_m$ (5)

where $r = < w_m, w_m > = \frac{m}{1-m}$ as above and all the components of the tangles are colored by $\widetilde{V}_{\nu_\alpha^0}$.

To compute the coefficients in these two relations, one can consider the highest weight vector v_+ and the lowest weight vector v_- of $\widetilde{V}_{\nu_\alpha^0}$. Their weight are respectively αw_m and $(\alpha + m - 1)w_m$. The vector v_+ is even but the parity of v_- is the parity of m. From the character formula one can also see that $v_+ \otimes v_+$ is a highest weight vector of $\widetilde{V}_m$ and $v_- \otimes v_-$ is a lowest weight vector of $\widetilde{V}_0$. So the two skein relations can be checked by evaluating the corresponding morphisms of $\widetilde{V}_{\nu_\alpha^0} \otimes \widetilde{V}_{\nu_\alpha^0}$ on $v_+ \otimes v_+$ and $v_- \otimes v_-$. Now the quasi-R-matrix acts by 1 on these two vectors, so the positive braiding is simply given by

F  $(v_+ \otimes v_+) = q^{<\alpha w_m, \alpha w_m>} v_+ \otimes v_+$

F  $(v_- \otimes v_-) = (-1)^m q^{<(\alpha+m-1)w_m, (\alpha+m-1)w_m>} v_- \otimes v_-$

where the $(-1)^m$ sign comes from the parity of v_-. The negative braiding acts by the inverse values and the skein relations (4) and (5) are consequences of this.

The skein relation (4) implies that M' satisfies the axiom of the Conway identity.

We now use (5) to check the modified doubled axiom for the oriented link L with its k ordered components colored by $\widetilde{V}_{\nu_{\alpha_i}^0}$, $i = 1 \cdots k$. We choose a framing on L so that $lk_{ii} = 0$. Hence $L^\pm$ can be obtain from L by replacing L_i by two parallel copies modified in a small ball as in the left hand side of (5). This gives

$$q^{2m\alpha_i - r\alpha_i^2} F'(L^+) - q^{-2m\alpha_i + r\alpha_i^2} F'(L^-) = (q^{2m\alpha_i} - q^{-2m\alpha_i}) F'(\overline{L})$$

where the j-th component of L^+, L^- and $\overline{L}$ is colored by α_j except the i-th component of $\overline{L}$ which is colored by $2\alpha_i$. Now as $lk_{ij}(L^\pm) = lk_{ji}(L^\pm) = 2lk_{ij}(L)$ for $j \neq i$ and $lk_{ii}(L^\pm) = \pm 1$, the framing correction of (2) gives

$$t^{\alpha_i} M'(L^+)(t^{\alpha_1}, \ldots, t^{\alpha_k}) - t^{-\alpha_i} M'(L^-)(t^{\alpha_1}, \ldots, t^{\alpha_k}) =$$
$$\left(\prod_{j \neq i} t^{lk_{ij}\alpha_j} \right) (t^{2\alpha_i} - t^{-2\alpha_i}) M'(L)(t^{\alpha_1}, \cdots, t^{\alpha_{i-1}}, t^{2\alpha_i}, t^{\alpha_{i+1}}, \cdots, t^{\alpha_k})$$

where $t = q^m$. Hence M' is the Conway function. □

2. The Hecke Category $\mathcal{H}$

In this section we introduce the Hecke category and consider its relations with the ribbon category of $U_h\mathfrak{sl}(m|n)$-module. We fix an integer $N \geq 3$.

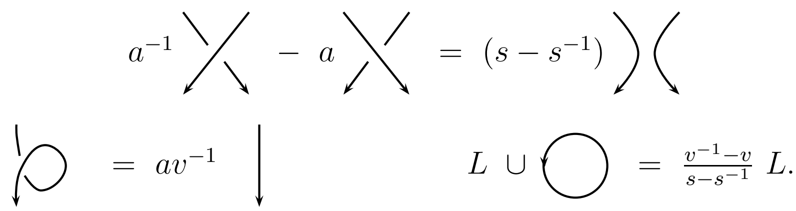
2.1. **Definition of the Hecke category.** Consider the quotient field K of the ring of polynomials in three variable a, s and v. Let $[k]_s = \dfrac{s^k - s^{-k}}{s - s^{-1}}$ for $k \in \mathbb{Z}$ and $[k]_s! = [k]_s [k-1]_s \cdots [1]_s$ for $k \in \mathbb{N}$. Let R be the sub-ring of K defined by

$$R = \mathbb{Q}\left[a^{\pm 1}, s^{\pm 1}, v^{\pm 1}, \frac{v - v^{-1}}{s - s^{-1}}, ([N-1]_s!)^{-1} \right]$$

We first briefly recall the definition of the R-linear category of framed oriented tangles $\mathcal{T}$. Let $\mathbf{D}^2 = \{z \in \mathbb{C} : |z| \leq 1\}$ be the standard disc in $\mathbb{C} \simeq \mathbb{R}^2$ and let M be the free associative monoid in two symbol $\{+;-\}$. To $(\alpha, \beta) \in M^2$ we associate the oriented 0-dimensional submanifold $\{(\frac{0}{p}, 0), \dots, (\frac{p-1}{p}, 0)\} \cup \{(\frac{0}{q}, 1), \dots, (\frac{q-1}{q}, 1)\} \subset \mathbf{D}^2 \times [0, 1]$ where p and q are the respective length of α and β. This set of points is oriented according to α and the opposite of β.

The set of object of $\mathcal{T}$ is M. The set of morphisms from α to β is the R-linear space span by isotopy class of framed oriented tangles T in $\mathbf{D}^2 \times [0, 1]$ such that the orientation of the boundary $\partial T = T \cap \mathbf{D}^2 \times \{0; 1\}$ is given by (α, β) and the framing on a point of ∂T is given by the vector $(i, 0)$. We denote this set of morphism by $\mathcal{T}(\alpha, \beta)$. The composition of $T_1 \in \mathcal{T}(\alpha, \beta)$ with $T_2 \in \mathcal{T}(\beta, \gamma)$ is obtained by gluing the pairs $(\mathbf{D}^2 \times [0, 1], T_1)$ and $(\mathbf{D}^2 \times [0, 1], T_2)$ where $(\mathbf{D}^2 \times \{0\}, \beta)$ is identified with $-(\mathbf{D}^2 \times \{1\}, -\beta)$. The tensor product for objects comes from the monoidal structure of M and the tensor product of morphisms is induced by trivial embeddings $\mathbf{D}^2 \times [0, 1] \amalg \mathbf{D}^2 \times [0, 1] \hookrightarrow \mathbf{D}^2 \times [0, 1]$.

We follow [3] to define a framed version of the Hecke category which we denote by $\mathcal{H}$: $\mathcal{H}$ is the quotient of $\mathcal{T}$ by the HOMFLY-PT skeins relations



We denote $F_{\mathcal{H}}$ the quotient functor from $\mathcal{T}$ to $\mathcal{H}$.

2.2. **Relation with $U_q\mathfrak{sl}(m|n)$.** Let V be the standard representation of $\mathfrak{g} = \mathfrak{sl}(m|n)$ of dimension $m + n$ (which is an atypical irreducible module). It is a well known fact that $F_{m|n}$ restricted to tangles whose components are all colored by $\widetilde{V}$ satisfies a HOMFLY-PT-type skein relation. More precisely, let $F_V : \mathcal{T} \longrightarrow \mathcal{T}_{U_h(\mathfrak{g})}$ be the functor that colors each component of a tangle with $\widetilde{V}$. For $\delta \in \mathbb{Z}^*$, let $\psi_\delta : R \longrightarrow \mathbb{Q}[q^{\pm 1/\delta}, [N-1]_q!^{-1}] \subset \mathbb{Q}[[h]]$ be the ring morphism defined by

$$\psi_\delta(s) = q, \quad \psi_\delta(v) = q^{-\delta} \quad \text{and} \quad \psi_\delta(a) = q^{-1/\delta}.$$

Proposition 2.1. *There exists an unique monoidal functor $\overline{F_{m|n}} : \mathcal{H} \longrightarrow \mathrm{Mod}_{U_h(\mathfrak{g})}$ such that the following diagram is commutative*

$$\begin{tikzcd} \mathcal{T} \arrow[r, "F_V"] \arrow[d, "F_{\mathcal{H}}"] & \mathcal{T}_{U_h(\mathfrak{g})} \arrow[d, "F_{m|n}"] \\ \mathcal{H} \arrow[r, "\overline{F_{m|n}}"] & \mathrm{Mod}_{U_h(\mathfrak{g})} \end{tikzcd} .$$

Furthermore, $\overline{F_{m|n}}$ is R-linear in the following way:

$$\forall x \in R, \ \forall T \in \mathrm{Mor}(\mathcal{H}), \ \overline{F_{m|n}}(xT) = \psi_{m-n}(x)\overline{F_{m|n}}(T).$$

2.3. **Framing and grading.** There is a little difference between the framed and unframed version of HOMFLY-PT. One recovers the unframed skein relations by taking $a = v$. But this quotient makes the statement of Proposition 2.1 and the cabling process less natural. So we prefer to use a framed version and introduce a "framing-degree" (f-deg) to make the correspondence more clear.

The category $\mathcal{T}$ possess a natural $\mathbb{Z}$-grading: if T is a tangle define f-deg$(T) \in \mathbb{Z}$ as the total algebraic number of crossing of T. For the ring elements, we consider the $\mathbb{Z}$-grading given by f-deg$(a^{\pm 1}) = \pm 1$, f-deg$(v^{\pm 1}) =$ f-deg$(s^{\pm 1}) = 0$. Thus the spaces of morphism of $\mathcal{T}$ are $\mathbb{Z}$-graded vector spaces. One can easily check that the HOMFLY-PT relations are homogeneous and thus $\mathcal{H}$ inherits this $\mathbb{Z}$-grading. If L is a framed link with $f =$ f-deg(L) then $F_{\mathcal{H}}(L)$ is an element of $a^f\mathbb{Z}\left[s^{\pm 1}, v^{\pm 1}, \frac{v-v^{-1}}{s-s^{-1}}\right]$.

3. The colored HOMFLY-PT polynomials dominate $M^c_{\mathfrak{sl}(m|n)}$

3.1. **Idempotents of the Hecke algebra.** Let $r \in \mathbb{N}^*$. The algebra $\mathcal{H}(+^r, +^r)$ is isomorphic to the Hecke algebra H_r of type A. The simple subfactors of $H_r \otimes K$ are indexed by Young diagrams λ of size $|\lambda| = r$. In other words, one has $H_r \subset H_r \otimes K \simeq \bigoplus_{|\lambda|=r} H_\lambda$ where H_λ is isomorphic to the ring of $d(\lambda)$ by $d(\lambda)$ matrices over K and $d(\lambda)$ is the integer $r!$ divided by the product of the hook lengths of cells of λ (we say that a cell $c = (i, j) \in \mathbb{N}^* \times \mathbb{N}^*$ is in λ if $j \le \lambda_i$. Then the hook length of c is $hl(c) = \lambda_i + \lambda'_j - i - j + 1$ with λ' the conjugate partition of λ).

Following [3], we use the notation $[hl(\lambda)]_s$ for the product over all cells c of λ of their quantum hook-lengths: $[hl(\lambda)]_s = \prod_{c\in\lambda} [hl(c)]_s$. We say that λ is R-admissible if $[hl(\lambda)]_s$ is invertible in R. Then the projector on the simple subfactor of type λ is realized by multiplication with a central idempotent of $c_\lambda \in H_r$ (c_λ is the sum of the $d(\lambda)$ minimal orthogonal idempotents of type λ constructed in [1, 3]). Moreover, $c_\lambda H_r$ is isomorphic to the ring of $d(\lambda)$ by $d(\lambda)$ matrices over R. For any R-admissible diagram λ, we choose a minimal idempotent y_λ of type λ (i.e. $y_\lambda \in c_\lambda H_r$). This choice is unique up to conjugation and f-deg$(y_\lambda) = 0$ (see [1, 3]).

3.2. **Classical limit.** Fix $\delta \in \mathbb{Z}^*$. We consider the surjective morphism of $\mathbb{Q}$-algebras π from H_r to the group ring of the permutation group $\mathbb{Q}[\mathfrak{S}(r)]$ given by:

$$\pi(a) = 1, \quad \pi(v) = 1, \quad \pi(s) = 1, \quad \pi\left(\frac{v - v^{-1}}{s - s^{-1}}\right) = -\delta.$$

Then $\pi(y_\lambda)$ is a minimal idempotent of the simple factor of $\mathbb{Q}[\mathfrak{S}(r)]$ of type λ. We now use the super version of the Schur functors "$\mathsf{S}^\lambda V$" in the following proposition which is the quantum analog of a theorem of Sergeev ([17] Theorem 2):

Proposition 3.1. *$\overline{F_{m|n}}(y_\lambda)$ is a minimal projection on an irreducible $U_h(\mathfrak{g})$-module $\widetilde{\mathsf{S}^\lambda V} \subset \widetilde{V}^{\otimes r}$. Write $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ (with $\lambda_i \le \lambda_{i+1}$ and $\lambda_i = 0$ for $i >> 0$) and denote by $\lambda' = (\lambda'_1, \lambda'_2, \lambda'_3, \dots)$ the conjugate partition of λ and $\lambda''_i = \max(\lambda'_i - m, 0)$. Then,*

1. *if $\lambda_{m+1} > n$ then $\widetilde{\mathsf{S}^\lambda V} = 0$.*
2. *If $\lambda_{m+1} \le n$ then $\widetilde{\mathsf{S}^\lambda V}$ is a deformation of an irreducible $\mathfrak{g}$-module with highest weight $(\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \dots, \lambda_{m-1} - \lambda_m, \lambda_m - \lambda''_1, \lambda''_1 - \lambda''_2, \lambda''_2 - \lambda''_3, \dots, \lambda''_{n-1} - \lambda''_n)$.*

(3) *In particular if* $\lambda_{m+1} \leq n$ *and* $\lambda_m \geq n$ *then* $\widetilde{V}^\lambda$ *has highest weight* $(\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \ldots, \lambda_{m-1} - \lambda_m, \lambda_m - \lambda'_1 + m, \lambda'_1 - \lambda'_2, \lambda'_2 - \lambda'_3, \ldots, \lambda'_{n-1} - \lambda'_n)$

Proof. Recall that C the functor "classical limit" is obtained by sending h to zero:

$$C : \mathrm{Mod}_{U_h(\mathfrak{g})} \longrightarrow \mathrm{Mod}_{\mathfrak{g}} \quad \text{given by} \quad \widetilde{V} \mapsto \widetilde{V}/(h\widetilde{V}).$$

In the following, we will identify the $\mathfrak{g}$-module V with $C(\widetilde{V})$. The representation of $H_r \simeq \mathcal{H}(+^r, +^r)$ given by $C \circ \overline{F_{m|n}}$ factor as $\rho \circ \pi$ where $\pi : H_r \longrightarrow \mathbb{Q}[\mathfrak{S}(r)]$ is the projection of H_r onto the group ring of the permutation group $\mathfrak{S}(r)$ and ρ is the super representation of $\mathfrak{S}(r)$ in $V^{\otimes r}$ (this is true because the universal R-matrix of $U_h(\mathfrak{g})$ is 1 modulo h).

Since $\pi(y_\lambda)$ is a minimal idempotent of the simple factor of $\mathbb{Q}[\mathfrak{S}(r)]$ of type λ, $\rho \circ \pi(y_\lambda)$ is a projection on an irreducible $\mathfrak{g}$-submodule $\mathsf{S}^\lambda V$ of $V^{\otimes r}$ (see [17]). Hence $\overline{F_{m|n}}(y_\lambda)$ is a projection on a $U_h(\mathfrak{g})$-module $\widetilde{\mathsf{S}^\lambda V} \subset \widetilde{V}^{\otimes r}$ whose classical limit is $\mathsf{S}^\lambda V$. Since $\mathsf{S}^\lambda V$ is irreducible it follows that $\widetilde{\mathsf{S}^\lambda V}$ is irreducible. The theorem then follows from the description of the $\mathfrak{gl}(m|n)$-module $\mathsf{S}^\lambda V$ made by Sergeev ([17] Theorem 2). $\square$

If $L = (L_1 \cup \cdots \cup L_k)$ is a framed oriented link with k components, and $\lambda^* = (\lambda^1, \ldots, \lambda^k)$ is a k-tuple of R-admissible Young diagrams, one can define the λ^*-satellite of L, denoted $\mathrm{sat}_{\lambda^*}(L)$, as the linear combination of links obtained by replacing a regular neighborhood of L_i by a torus that contains the braid closure of y_{λ^i}, for each $1 \leq i \leq k$. Suppose now that L is the closure of the (unique up to isotopy) $(1,1)$-tangle $\mathrm{cut}(L, L_i) \in \mathcal{T}(+,+)$ obtained by opening the component L_i. Then $\mathrm{cut}(L, L_i)$ inherits the coloration λ^* and one can define the λ^*-cable of $\mathrm{cut}(L, L_i)$ as $\mathrm{sat}_{\lambda^*}(\mathrm{cut}(L, L_i)) \in \mathcal{T}(+^{|\lambda^i|}, +^{|\lambda^i|})$.

A simple count shows that f-deg(cut(L, L_i)) = f-deg(L) and

$$\text{f-deg}(\mathrm{sat}_{\lambda^*}(\mathrm{cut}(L, L_i))) = \text{f-deg}(\mathrm{sat}_{\lambda^*}(L)) = {}^t|\lambda^*|.lk(L).|\lambda^*|$$

where $lk(L)$ is the linking matrix of L and $|\lambda^*|$ is the vector column $(|\lambda^1|, |\lambda^2|, \ldots, |\lambda^k|)$. As y_{λ^i} is an idempotent in $\mathcal{H}$,

$$F_{\mathcal{H}}(\mathrm{sat}_{\lambda^*}(\mathrm{cut}(L, L_i))) \circ y_{\lambda^i} = y_{\lambda^i} \circ F_{\mathcal{H}}(\mathrm{sat}_{\lambda^*}(\mathrm{cut}(L, L_i))) = F_{\mathcal{H}}(\mathrm{sat}_{\lambda^*}(\mathrm{cut}(L, L_i))).$$

Now since y_{λ^i} is minimal, we have $F_{\mathcal{H}}(\mathrm{sat}_{\lambda^*}(\mathrm{cut}(L, L_i))) = x.y_{\lambda^i}$ for some scalar $x \in R$. Thus, using the convention given at the beginning of Section 1 we have $x = \langle F_{\mathcal{H}}(\mathrm{sat}_{\lambda^*}(\mathrm{cut}(L, L_i))) \,:\, y_{\lambda^i} \rangle$.

Definition 3.2. *For any k-component framed oriented colored link L whose coloring is given by a k-tuple of R-admissible Young diagrams λ^* we define*

$$H(L, \lambda^*) = F_{\mathcal{H}}(\mathrm{sat}_{\lambda^*}(L)), \qquad H'(L, \lambda^*, L_i) = \langle F_{\mathcal{H}}\left(\mathrm{sat}_{\lambda^*}(\mathrm{cut}(L, L_i))\right) \,:\, y_{\lambda^i} \rangle\,.$$

We call H the colored HOMFLY-PT polynomial and H' the reduced colored HOMFLY-PT polynomial.

Also remark that $H(L, \lambda^*) = H'(L, \lambda^*, L_i).H(\text{unknot}, \lambda^i)$.

Remark 3.3. (1) *As shown in* [1] *the twist acts on y_λ by the scalar*

$$\theta_\lambda = a^{|\lambda|^2} v^{-|\lambda|} s^{2n(\lambda)}$$

where $n(\lambda) = \sum_{(i,j)\in\lambda} j - i = \sum i(\lambda_i - \lambda'_i)$ (the first sum runs over the coordinates of the cells in the young diagram λ). So one can re-normalize the colored HOMFLY-PT polynomial to get an invariant of colored oriented link (not framed). For example, if

all the component of a framed link are colored with the same λ then let w be the total linking number of L (the algebraic number of crossing of L) and so, $\theta_\lambda^{-w} H(L,\lambda^*)$ is an invariant of the underlying oriented link.
We can define similarly with the same correction an unframed version of H'. Remark that these two unframed invariants are computed with framing degree 0 elements of $\mathcal{H}$ and so they are elements of $\mathbb{Q}(v,s)$.

(2) *In the case of a knot (that is if there is only one component), it follows from [14] that $H'(L,\lambda,L)$ is always a Laurent polynomial in the variables a,v,s but this is not true for links with several components, even for the Hopf link (see [13]).*

Proposition 3.4. *For any framed oriented link L with k components and any k-tuple $\lambda^* = (\lambda^1,\dots,\lambda^k)$ of R-admissible Young diagrams, one has*

(1) $\psi_{m-n}(H(L,\lambda^*)) = \left\langle F_{m|n}(L;\widetilde{V}^{\lambda^*}) \,:\, \mathrm{Id}_{\mathbb{C}[[h]]}\right\rangle$

(2) $\psi_{m-n}(H'(L,\lambda^*,L_i)) = \left\langle F_{m|n}\left(\mathrm{cut}(L,L_i);\widetilde{V}^{\lambda^*}\right) \,:\, F_{m|n}\left(y_{\lambda^i}\right)\right\rangle$

(3) *So, if $\mathsf{S}^{\lambda^i}V$ is typical then*

$$\psi_{m-n}\left(\frac{H(L,\lambda^*)}{H(unknot,\lambda^i)}\right) = \psi_{m-n}\left(H'(L,\lambda^*,L_i)\right) = \frac{F'_{m|n}((L;\widetilde{\mathsf{S}^{\lambda^*}V}))}{\mathsf{d}(\widetilde{\mathsf{S}^{\lambda^i}V})}$$

where $(L;\widetilde{\mathsf{S}^{\lambda^*}V})$ denote the link $L \in \mathcal{T}_{U_h(\mathfrak{g})}$ with its i^{th} component colored by $\widetilde{\mathsf{S}^{\lambda^i}V}$, for all $1 \le i \le k$.

Proof. The proof of (1) follows from the commutative diagram in Proposition 2.1. For (2) we have $F_{\mathcal{H}}(\mathrm{sat}_{\lambda^*}(\mathrm{cut}(L,L_i))) = x.y_{\lambda^i}$ for some $x \in R$. Proposition 2.1 implies that $\overline{F_{m|n}}(x.y_{\lambda^i}) = \psi_{m-n}(x)\overline{F_{m|n}}(y_{\lambda^i})$. Now Proposition 3.1 states that $\overline{F_{m|n}}(y_\lambda)$ is a minimal projection on an irreducible $U_h(\mathfrak{g})$-module $\widetilde{\mathsf{S}^\lambda V} \subset \widetilde{V}^{\otimes r}$. Thus, the commutative diagram of Proposition 2.1 implies (2). Finally, (3) follows from (2) and the definition of $F'_{m|n}$, since all three quantities in the statement correspond to the scalar associated to the (1,1)-tangle whose closure is L. □

Remark that the first two statements of the proposition are still valid when $n=0$, i.e. when $\mathfrak{g}$ is the Lie algebra $\mathfrak{sl}(m)$.

Corollary 3.5. *Let L be an oriented link with k ordered components. The multivariable invariant $M^c_{\mathfrak{sl}(m|n)}(L)$ associated with $\mathfrak{sl}(m|n)$ and $c \in \mathbb{N}^{m+n-2}$ is determined by the infinite family of colored HOMFLY-PT polynomials of L.*

Proof. First remark that for fixed c there is finitely many α such that $\widetilde{V}_{\nu^c_\alpha}$ is an atypical module. So there exists a $K_c \in \mathbb{N}$ such that, for all $a \in \mathbb{N}$, with $a \ge K_c$ we have $\widetilde{V}_{\nu^c_a}$ is typical. Now, for $i=1,...,k$, let $a_i \in \mathbb{N}$ such that $a_i \ge K_c$. Then $M^c_{\mathfrak{sl}(m|n)}(L)(q,q^{a_1},\dots,q^{a_k})$ is determined by the linking matrix of L and by $F'_{m|n}(L;(\widetilde{V}_{\nu^c_{a_1}},\dots,\widetilde{V}_{\nu^c_{a_k}}))$ as all the modules $\widetilde{V}_{\nu^c_{a_i}}$ are typical.

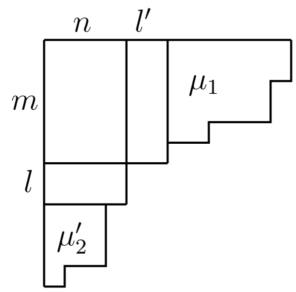
Since $M^c_{\mathfrak{sl}(m|n)}(L)(q,q_1,\dots,q_k)$ (resp. $M^c_0(q,q_1)M^c_{\mathfrak{sl}(m|n)}(L)(q,q_1)$ if $k=1$, where $M^c_0(q,q_1)$ is defined in the Appendix) is a Laurent polynomial, it is determined by the infinite family of polynomials $M^c_{\mathfrak{sl}(m|n)}(L)(q,q^{a_1},\dots,q^{a_k})$ for $a_i > K_c$. Hence to prove the corollary it suffices

to show that when $a_i \geq K_c$ the invariant $F'_{m|n}(L; (\widetilde{V}_{\nu^c_{a_1}}, \ldots, \widetilde{V}_{\nu^c_{a_k}}))$ can be computed from the colored HOMFLY-PT polynomial of L.

To prove this let $a \in \mathbb{N}$, with $a \geq K_c$ and choose $(l', l) \in \mathbb{N}^2$ such that $m + n + l' - (m + l + \sum_{j=m}^{m+n-2} c_j) = a$, where $c = (c_1, ..., c_{m+n-2})$. Let $\mu_1 = (\sum_{j=i}^{m-1} c_j)_{i=1\cdots m}$ and $\mu_2 = (\sum_{j=m-1+i}^{m+n-2} c_j)_{i=1\cdots n}$. We construct the partition

$$\lambda = \left[\left(\underbrace{n + l', \ldots, n + l'}_{m} \right) + \mu_1 \right] \cup \underbrace{n, \ldots, n}_{l} \cup \mu_2'.$$

where μ_2' is the conjugate partition of μ_2

i.e. $\lambda =$ 

Then one has $\lambda_i - \lambda_{i+1} = c_i$ for $i = 1, \ldots, m-1$ and $\lambda'_i - \lambda'_{i+1} = c_{m-1+i}$ for $i = 1, \ldots, n-1$. Furthermore, $\lambda_m = n + l' \geq n$ and $\lambda_{m+1} \leq n$. Thus the module $\widetilde{\mathsf{S}^\lambda V}$ is isomorphic to $\widetilde{V}_{\nu^c_a}$. This with (3) of Proposition 3.4 completes the proof. $\square$

4. Rank-Level duality in $\mathcal{H}$ and Kashaev's invariant

Throughout this section, N is an integer greater than 2 and $\xi = \mathrm{e}^{\sqrt{-1}\pi/N}$ is the primitive $2N$ root of 1. In [2] an invariant of links L whose components are colored by complex numbers is defined. This invariant come from the Reshetikhin-Turaev functor of tangles colored by $U_q(\mathfrak{sl}(2))$-finite dimensional modules when q is a root of unity (see [16, 9]). More precisely, each component of L shall be colored with the nilpotent representation of $U_q(\mathfrak{sl}(2))$ with highest weight the complex number associated to it.

Definition 4.1. *Let ADO_N be the ordered oriented link invariant defined for a link L with k components and linking matrix lk_{ij} by*

$$ADO_N(L)(a_1, \ldots, a_k) = q^{-\sum lk_{ij} a_i (a_j + 2 - 2N)/2} \Phi^N_L(a_1, \ldots, a_k)$$

where $q = \mathrm{e}^{\sqrt{-1}\pi/N}$, $a_i \in \mathbb{C}$ and Φ^N_L is the framed ordered link invariant given in [16].

The invariant ADO_N is a slightly modified version of the analogous invariant constructed in [2]. Contrary to [2, 15], here we require that the two dimensional representation of $U_q(sl_2)$ has weight 1.

Theorem 4 (H. and J. Murakami, [15], Theorems 4.9 and 2.1)**.** *For all $N \geq 2$ and for any link L*

$$K_N(L) = J_N(L)_{|q=\xi} = ADO_N(L)(N-1, \ldots, N-1).$$

The colored Jones polynomial can be computed as the $\mathfrak{sl}(2)$ specialization of the colored HOMFLY-PT polynomial. The representation of $\mathfrak{sl}(2)$ used by Murakami is the $N-1^{\text{th}}$ symmetric power of the standard representation of $\mathfrak{sl}(2)$, i.e. the representration $\mathsf{S}^{[N-1]}V$

where $[N-1]$ stands for the partition of $N-1$ consisting in only one part (or the young diagram with only one row) and V is the standard representation of dimension 2. Hence, Theorem 4 can be restated as follows.

Proposition 4.2. *Let L be an ordered oriented link with k components. As before we fix a framed representant of L and we still denote it by L. Then*

$$\psi_2\big((\theta_{[N-1]})^{-w}\, H'(L;(\underbrace{[N-1],\dots,[N-1]}_{k});L_i)\big)\Big|_{q=\xi} = K_N(L)$$

where $[N-1]$ stands for the young diagram corresponding to the one part partition of $N-1$, w is the sum of all the linking numbers of L and $(\theta_{[N-1]})^{-w}$ is the framing correction as given in Remark 3.3 (1)*, i.e. $\theta_{[N-1]} = a^{(N-1)^2}v^{-(N-1)}s^{(N-1)(N-2)}$.*

Proof. We can apply statement (2) of Proposition 3.4 to $\mathfrak{sl}(2)$. As we said, the $N-1^{\text{th}}$ symmetric power of the standard representation of $\mathfrak{sl}(2)$ is the N-dimensional irreducible $\mathfrak{sl}(2)$-module whose deformation is used to compute $J_N(L)$. We just need to make the standard correction of H' so that it is a link invariant (i.e. framing independent). □

We now introduce a symmetry of $\mathcal{H}$ that leads to the rank-level duality of modular categories of type A.

Definition 4.3. *Let $\Theta : R \longrightarrow R$ be the ring involution defined by*

$$\Theta(s) = s^{-1} \quad \Theta(a) = -a \quad \Theta(v) = -v$$

We extend Θ to an involutive functor $\Theta : \mathcal{T} \longrightarrow \mathcal{T}$ that fixes the tangles.

Proposition 4.4. *The involution Θ induces a well defined involutive $\mathbb{Q}$-linear endo-functor of $\mathcal{H}$ such that $\Theta(y_\lambda)$ is a minimal projector of type λ' and thus for any framed link with k components,*

$$\Theta(H(L,\lambda^*)) = H(L,\lambda^{*\prime}) \quad \textit{and} \quad \Theta(H'(L,\lambda^*,L_i)) = H'(L,\lambda^{*\prime},L_i)$$

where $\lambda^{*\prime}$ is the k-tuple of the conjugate young diagrams of λ^*.

Proof. First, Θ is well defined on $\mathcal{H}$ because Θ fixes the HOMFLY-PT skein relations. In [1, 3] the symetrizer and antisymetriser of H_r are defined respectively by:

$$f_r = \frac{1}{[r]_s!}s^{-r(r-1)/2}\sum_{\pi\in\mathfrak{S}(r)}(a/s)^{-l(\pi)}w_\pi$$

$$\text{and} \quad g_r = \frac{1}{[r]_s!}s^{r(r-1)/2}\sum_{\pi\in\mathfrak{S}(r)}(-as)^{-l(\pi)}w_\pi$$

where w_π are braids of length $l(\pi)$ (the positive permutation braids). Clearly $\Theta(f_r) = g_r$ and $\Theta(g_r) = f_r$. Now if $\lambda = (\lambda_1 = l, \lambda_2, \dots, \lambda_k)$ is R-admissible, then so is its conjugate partition $\lambda' = (\lambda'_1 = k, \lambda'_2, \dots, \lambda'_l)$ (because $[hl(\lambda)]_s = [hl(\lambda')]_s$). Furthermore, $c_\lambda H_r$ is the intersection of the two two-side ideals of H_r generated by $f_{\lambda_1}\otimes\cdots\otimes f_{\lambda_k}$ and $g_{\lambda'_1}\otimes\cdots\otimes g_{\lambda'_l}$. Hence $\Theta(c_\lambda H_r) = c_{\lambda'}H_r$ and so $\Theta(y_\lambda)$ is a minimal idempotent of type λ'. □

Lemma 4.5. *For any element $x \in \mathbb{Q}(s)[v^{\pm}]$ without a pole at $s = \xi$, we have*

$$\psi_\delta \circ \Theta(x)\big|_{q=\xi} = \psi_{N-\delta}(x)\big|_{q=\bar{\xi}} \tag{6}$$

where $\bar{\xi}$ is complex conjugation of ξ. Consequently, if the k-tuple of young diagrams λ^* satisfy $|\lambda^i| < N$ for $i = 1 \cdots k$, then for a framed link with k components colored by λ^*, one has

$$\psi_\delta((a^{-1}v)^w H'(L, \lambda^*, L_i))\big|_{q=\xi} = \psi_{N-\delta}((a^{-1}v)^w H'(L, \lambda'^*, L_i))\big|_{q=\bar{\xi}}$$

where $w = {}^t|\lambda^*|.lk(L).|\lambda^*| = \text{f-deg}(H'(L, \lambda^*, L_i))$

Proof. Equation (6) follows from a direct calculation. If all young diagrams have size less than N, there will be no pole at $s = \xi$ in the expression of the associated idempotents y_{λ^i}. The factor $(a^{-1}v)^w$ is invariant by Θ and $(a^{-1}v)^w H'(L, \lambda^*, L_i) \in \mathbb{Q}(s)[v^{\pm 1}]$. Then the last statement of the lemma follows from Equation (6). $\square$

Lemma 4.6. *Let V be the standard representation of $\mathfrak{sl}(N-1|1)$ and $[1^{,N-1}] = [1, 1, \dots, 1]$ be the $N-1$ parts partition of $N-1$ (or the young diagram with only one column). Then $\mathsf{S}^{[1^{,N-1}]}V = \Lambda^{N-1}V = V_1^0$ is the typical representation of $\mathfrak{sl}(N-1|1)$ with highest weight $(0, \dots, 0, 1)$. Its modified quantum dimension d has the following properties*

$$\mathsf{d}_{\mathfrak{sl}(N-1|1)}(\widetilde{V}_1^0) = \prod_{i=1}^{N-1} (q^i - q^{-i})^{-1} = \{N-1\}!^{-1}$$

$$\mathsf{d}_{\mathfrak{sl}(N-1|1)}(\widetilde{V}_1^0)\Big|_{q=\xi} = \frac{\mathrm{e}^{-\sqrt{-1}(N-1)\pi/2}}{N}.$$

Proof. The first statement follows from Proposition 3.1. The formulas for d are taken from [8]. $\square$

Theorem 5. *Let L be an oriented link. Let $M^0_{\mathfrak{sl}(N-1|1)}$ as defined in* [8] *and $\xi = \mathrm{e}^{\sqrt{-1}\pi/N}$. Then*

$$K_N(L) = N\,\mathrm{e}^{\sqrt{-1}(N-1)\pi/2}\, M^0_{\mathfrak{sl}(N-1|1)}(L)(\bar{\xi}, \bar{\xi}, \dots, \bar{\xi}) = LG^{N-1|1}(L)(\xi, \bar{\xi}).$$

Proof. Let L be an ordered link. Fix a framed representant L_f of L. We consider the following three colorings of L_f. Let L' be L_f with all its components colored by the module $\widetilde{V}_{\nu_1^0}$. Let L'' (resp. L''') be L_f colored by the young diagram corresponding to the $N-1$ parts (resp. to the one part) partition of $N-1$. Let w is the sum of all the linking numbers of L_f. Let $\theta_{[1^{,N-1}]}$ and $\theta_{[N-1]}$ be the values of the twist for the colored HOMFLY-PT given in Remark 3.3 (1) and let θ_1^0 be the value of the twist on $\widetilde{V}_{\nu_1^0}$.

With this notation we have

$$\begin{aligned}
N\,\mathrm{e}^{\sqrt{-1}(N-1)\pi/2}\, M^0_{\mathfrak{sl}(N-1|1)}(L)(\bar{\xi}, \bar{\xi}, \dots, \bar{\xi}) &= \mathsf{d}_{\mathfrak{sl}(N-1|1)}(\widetilde{V}_{\nu_1^0})^{-1}(\theta_0^1)^{-w} F'(L')\Big|_{q=\bar{\xi}} \\
&= \psi_{N-2}\left((\theta_{[1^{,N-1}]})^{-w} H'(L'')\right)\Big|_{q=\bar{\xi}} \\
&= \psi_2\left(\Theta((\theta_{[1^{,N-1}]})^{-w} H'(L'')\right)\Big|_{q=\xi} \\
&= \psi_2\left((\theta_{[N-1]})^{-w} H'(L''')\right)\Big|_{q=\xi} \\
&= K_N(L)
\end{aligned}$$

where the first equality follows from the definition of $M^0_{\mathfrak{sl}(N-1|1)}$ and Lemma 4.6, the second from Proposition 3.4 (3), the third from Lemma 4.5, the fourth from Proposition 4.4 and the last equality follows from Proposition 4.2. $\square$

We are lead to the following conjecture.

Conjecture 4.7. *Let* $\xi = \mathrm{e}^{\sqrt{-1}\pi/N}$ *then* $M^0_{\mathfrak{sl}(N-1|1)}(L)(\xi, \xi^{a_1}, \dots, \xi^{a_c}) = ADO_N(a_1, \dots, a_c)$.

Appendix

In this appendix we set $\mathfrak{g} = \mathfrak{sl}(m, 1)$. In Lemma 3.2 of [8] we define two Laurent polynomials $M^c_0 \in \mathbb{Z}[q^{\pm 1}]$ and $M^c_1 \in \mathbb{Z}[q^{\pm 1}, q_1^{\pm 1}]$ and show that

$$\mathsf{d}(V_{\nu^c_\alpha}) = \frac{M^c_0(e^{h/2})}{M^c_1(e^{h/2}, e^{\alpha h/2})}$$

where $c \in \mathbb{N}^{m-1}, \alpha \in \mathbb{C}$. When $c = 0$ we have $M^0_1(q, q_1) = \prod_{i=0}^{m-1}(q_1 q^i - (q_1 q^i)^{-1})$ and $M^0_0(q) = 1$. Therefore, we have the following lemma.

Lemma 4.8. $\mathsf{d}(\widetilde{V}_{\nu^0_\alpha}) = \frac{1}{\prod_{i=0}^{m-1}(q^{\alpha+i} - q^{-(\alpha+i)})}$.

Now a direct calculation gives the following lemma.

Lemma 4.9. $M^0_1(\mathrm{e}^{\sqrt{-1}\pi/m}, q_1) = \mathrm{e}^{\sqrt{-1}(m-1)\pi/2}(q_1^m - q_1^{-m})$.

Let V_l be the $\mathfrak{g}$-modules determined by Equation 3. For $0 < l < m$ we have $V_l = V_{\nu^{c_l}_{2\alpha+m-i-1}}$ where c_l is the l-tuple with a 1 in the l slot and 0 everywhere else.

Lemma 4.10. *When* $q = e^{\sqrt{-1}\pi/m}$ *we have* $\mathsf{d}(V_l) = 0$ *for* $0 < l < m$.

Proof. Using the notation and results in the Appendix of [8] we have

$$\rho = \sum (m-i)\epsilon_i, \quad <\rho, \epsilon_i - \epsilon_j> = j - i, \quad <w_k, \epsilon_i - \epsilon_j> = \begin{cases} 1 & \text{if } i \leq k < j, \\ 0 & \text{else.} \end{cases}$$

These equalities and the fact that the Laurent polynomials M^c_0 and M^c_1 are equal to particular specializations of elements in the group ring of the weight lattice (see Lemma 3.2 of [8]) imply

$$M^{c_l}_0(q) = \frac{\{m\}!}{\{l\}!\{m-l\}!} = \frac{\prod_{i=1}^{l}\prod_{j=l+1}^{m}\{j-i+1\}}{\prod_{i=1}^{l}\prod_{j=l+1}^{m}\{j-i\}}.$$

Therefore, when $q = e^{\sqrt{-1}\pi/m}$ we have $\mathsf{d}(V_l) = M^{c_l}_0(q)/M^{c_l}_1(q, q^{2\alpha+m-l-1}) = 0$. $\square$

References

[1] A. K. Aiston, H. R. Morton - *Idempotents of Hecke algebras of type A*, J. Knot Theory Ramifications **7** (1998), no. 4, 463–487.

[2] Y. Akutsu, T. Deguchi, and T. Ohtsuki - *Invariants of colored links.* J. Knot Theory Ramifications **1** (1992), no. 2, 161-184.

[3] C. Blanchet - *Hecke algebras, modular categories and 3-manifolds quantum invariants.* Topology **39** (2000), no. 1, 193–223.

[4] D. De Wit, A. Ishii, J.R. Links - *Infinitely many two-variable generalisations of the Alexander-Conway polynomial.* Algebr. Geom. Topol. 5 (2005), 405–418.

[5] T. Deguchi - *Multivariable invariants of colored links generalizing the Alexander polynomial.* Proceedings of the Conference on Quantum Topology (Manhattan, KS, 1993) (River Edge, NJ), World Sci. Publishing, 1994, 67–85.

[6] N. Geer - *Etingof-Kazhdan quantization of Lie superbialgebras.* Advances Math. **207** (2006), no. 1, 1–38.

[7] N. Geer, B.Patureau-Mirand - *Multivariable link invariants arising from* $\mathfrak{sl}(2|1)$ *and the Alexander polynomial.* J. Pure Appl. Algebra 210 (2007), no. 1, 283–298. (math.GT/0601291).
[8] N. Geer, B.Patureau-Mirand - *Multivariable link invariants arising from Lie superalgebras of type I.* Preprint (math.GT/0609034).
[9] N. Geer, B.Patureau-Mirand, V. Turaev - *Modified quantum dimensions and re-normalized link invariants.* Preprint.
[10] V.G. Kac - *Lie superalgebras.* Advances Math. 26 (1977), 8–96.
[11] R. M. Kashaev - *A link invariant from quantum dilogarithm.* Modern Phys. Lett. A **10** (1995), no. 19, 1409–1418.
[12] J.R. Links, M. Gould - *Two variable link polynomials from quantum supergroups.* Lett. Math. Phys. **26** (1992), no. 3, 187–198.
[13] S.G. Lukac, H.R. Morton - *The HOMFLY polynomial of the Decorated Hopf Link* J. Knot Theory Ramif. 12 (2003) 395-416.
[14] H.R. Morton - *Integrality of Homfly (1,1)-tangle invariants* Algebraic and Geometric Topology, 7 (2007), 227-238.
[15] H. Murakami, J. Murakami - *The colored Jones polynomials and the simplicial volume of a knot.* Acta Math. 186 (2001), no. 1, 85–104.
[16] J. Murakami - *Colored Alexander Invariants and Cone-Manifolds.* Acta Math. 186 (2001), no. 1, 85–104.
[17] A.N. Sergeev - *Tensor algebra of the identity representation as a module over the Lie superalgebras* Gl(n, m) *and* $Q(n)$. Mat. Sb. (N.S.) 123(165) (1984), no. 3, 422–430.
[18] N.Y. Reshetikhin, V.G. Turaev - *Ribbon graphs and their invariants derived from quantum groups*, Comm. Math. Phys. **127** (1990), no. 1, 1–26.
[19] V.G. Turaev - *Reidemeister torsion in knot theory.* [J] Russ. Math. Surv. 41, No.1, 119-182 (1986). (English translation of Uspekhi Mat. Nauk 41 (1986), no. 1(247), 97–147, 240.)
[20] V.G. Turaev - *Quantum invariants of knots and 3-manifolds.* de Gruyter Studies in Mathematics, 18. Walter de Gruyter & Co., Berlin, (1994) 588 pp.

School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160
E-mail address: `geer@math.gatech.edu`

LMAM, Université de Bretagne-Sud, BP 573, F-56017 Vannes, France
E-mail address: `bertrand.patureau@univ-ubs.fr`